COMPACTNESS AND SEMI-CONTINUITY

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ABSTRACT

The purpose of this note is to point out that a recent result of Ceder yields easily converses to well known theorems of Wallace and Birkhoff and thus provides two new characterazations of compactness as well as specifying the class of spaces for which the theorems are true. A very slight extension of Ceder's theorem is also obtained, as well as new and simple proofs.

Let X be a non-empty topological space and P a partially ordered set. A function $f: X \to P$ is an upper-(lower-)semicontinuous function iff $x_\alpha \to x_0$ and eventually $f(x_\alpha) \geqq c(\leqq c)$ implies that $f(x_0) \geqq c$ $(\leqq c)$. The theorem of Birkhoff ([1], p. 63) asserts that *when X is compact,*

(B) *every upper-(lower-)semicontinuous function to a partially ordered set assumes a maximal (minimal) value.*

A quasi order $\leqq$ (reflexive and transitive) on a non-empty topological space X is upper-(lower-) semicontinuous iff for each $x \in X$, $\{y \in X \mid x \leqq y\}$ $(\{y \in X \mid y \leqq x\})$ is closed in X. The theorem of Wallace ([4]) asserts that *when X is compact,*

(*W*) *each upper-(lower-)semicontinuous quasi order on X has a maximal (minimal) element.*

(See [5] p. 146 for a proof.)

That (B) and (W) are equivalent is part of the folklore of the subject and is not difficult to verify. It will emerge from Ceder's theorem that they hold precisely on the class of compact spaces.

Let X and Y be non-empty topological spaces, $\mathfrak{P}_0(Y)$ $(\mathfrak{P}_c(Y))$ the colloction of non-empty (closed) subsets of Y. A set valued function $f: X \to P_0(Y)$ is an upper-(lower-)semicontinuous carrier iff for each open $U \subseteq Y$, $\{x \mid f(x) \subseteq U\}$

$$(\{x \mid f(x) \cap U \neq \phi\})$$

is open in X. Ceder's theorem ([2]) asserts that *a necessary and sufficient condition for X to be compact is that*

(C) *each upper-(lower-)semicontinuous carrier into the closed subsets of a T_1 space assumes a maximal (minimal) value with respect to set inclusion.*

It is possible to strengthen (C) by deleting the word closed without destroying the result if one restricts attention to the upper-semicontinuous case. Call this strengthened form (C′) A proof of the necessity of (C′) to compactness is essentially

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contained in the first part of the proof of Theorem 1 of [2], the basic idea being that a maximal element for a chain $\{f(x_i)\}$ of images is provided by the image $f(x_0)$ of any cluster point x_0 of the net $\{x_i\}$ giving rise to the chain. The sufficiency is immediate since (C′) implies the uppersemicontinuous case of (C).

A simple proof of the necessity of (C) can be had from the theorem of Birkhoff by noting that *each upper-(lower-)semicontinuous carrier is an upper-(lower-) semicontinuous function with respect to set inclusion as the partial order on* $\mathfrak{P}_0(Y)(\mathfrak{P}_c(Y))$ This fact also implies the sufficiency of (B) for compactness. More precisely it shows that (B) implies (C) which by Ceder's theorem implies compactness.

The upper-(lower-)semifinite topology on $\mathfrak{P}_0(Y)$ is obtained by taking the family of all sets of the form $\{A \mid A \subseteq U\}$ $(\{A \mid A \cap U \neq \phi\})$, where U is open in Y, as a subbasis. One sees immediately that $f: X \to \mathfrak{P}_0(Y)$ is an upper-(lower-)semicontinuous carrier iff f is continuous with respect to the upper-(lower-)semifinite topology on $\mathfrak{P}_0(Y)$ (see Michael, [3], p. 179).

Still another easy demonstration of the necessity of (C) can be had directly from the theorem of Wallace by noting that *set inclusion is an upper-(lower-) semicontinuous quasi order on* $\mathfrak{P}_c(Y)$ *with the upper-(lower-)semifinite topology* and that continuous images of compact sets are compact.

The sufficiency of (W) follows from its equivalence to (B). (It is also easy to verify directly that (W) implies (C).)

Hence we may assert that *each of the conditions* (B), (W), (C), (C′) *is necessary and sufficient for compactness.*

Also, Ceder's construction for the second part of the proof of his Theorem 1 can be modified to yield simple direct proofs of the sufficiency of (B) and (W). For example, to show that (B) does not hold in a non-compact space (for the lower-semicontinuous case), let $\{y_a\}_{a \in D}$ be a net in X with no cluster point. Following Ceder, the collection of nonempty open sets $V_b = X \setminus \{y_a \mid a \geqq b\}^-$ cover X. For each $x \in X$, let $f(x) = \{b \mid x \in V_b\} \subseteq D$. Suppose in X a net $\{x_\alpha\}$ converges to x_0 and eventually $f(x_\alpha) \subseteq C \subseteq D$. If $a \in f(x_0) \setminus C$, eventually $x_\alpha \in V_a$ contradicting $f(x_\alpha) \subseteq C$. Hence f is a lower-semicontinuous function into $\mathfrak{P}_0(D)$, partially ordered by set inclusion, which assumes no minimum.

References

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